THE THEORY OF THE THERMOHYDRODYNAMIC GAS LENS

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UDC 535.31:532.542

We find an analytical solution for the equation of the beam trajectory for arbitrary lens-like media of the type $n = n(y)$. We examine the optical properties of certain specific lens-type media.

Particular attention is being devoted at the present time to the development of lens-like media by a thermal process [1, 2]. Particular importance is ascribed to the development of the theory of thermodynamic gas lenses. The study of the behavior of a light beam in a gas lens is possible with very good approximation within the framework of geometric optics. Let us analyze the trajectory of a beam in a cylindrical gas lens with arbitrary axisymmetric temperature distribution. The distribution of the refractive index in such a lens is also axisymmetric, and we can therefore limit ourselves to the plane case. In Cartesian coordinates the equation for the trajectory has the form

$$y'' = \frac{1}{n}\left\{(1 + y'^2)\frac{\partial n}{\partial y} - y'(1 + y'^2)\frac{\partial n}{\partial x}\right\}. \tag{1}$$

We will assume – as is the case under real conditions – that the refractive index is a weak function of the x coordinate in the direction of beam propagation. The refractive index may be regarded in this case as an exclusive function of the radius $n = n(y)$. The literature contains approximate solutions of (1) for certain special cases [3, 4]. Here we find an exact solution for (1) for arbitrary lens-like media of the $n = n(y)$ type and we investigate the specific lens-like media. It is demonstrated that unlike the approximate solution, the exact solution of (1) yields qualitatively new results.

For arbitrary lens-like media of the $n = n(y)$ type Eq. (1) has the form

$$y'' = \frac{1}{n}(1 + y'^2)\frac{\partial n}{\partial y}. \tag{2}$$

Taking y as an independent variable and introducing the new function

$$u(y) = y'^2, \tag{3}$$

we bring Eq. (2) to the form of a linear first-order nonuniform equation, i.e.,

$$\frac{du}{dy} - 2\frac{1}{n}\frac{dn}{dy}u = 2\frac{1}{n}\frac{dn}{dy}, \tag{4}$$

whose over-all solution has the form

$$u = y'^2 = c_1 n^2(y) - 1. \tag{5}$$

Integrating (5) yields the beam trajectory

$$x - c_2 = \int \frac{dy}{\sqrt{c_1 n^2(y) - 1}}. \tag{6}$$

The arbitrary constants c_1 and c_2 are determined from the boundary conditions

$$y'|_{x=0} = y'_0, \quad y|_{x=0} = y_0. \tag{7}$$

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 1, pp. 145-149, July, 1969. Original article submitted August 2, 1968.

In addition, let us examine certain specifics from lens-like media. We know [6, 7] that in a tube with a uniform heat flow through the wall, in the presence of forced convection, given sufficiently large values of x we have a completely developed temperature profile, and namely:

$$\frac{T-T_0}{q\frac{R}{\lambda}} = \frac{4x}{\mathrm{Pr}\,\mathrm{Re}} + y^2 - \frac{1}{4}y^4 - \frac{7}{24}. \tag{8}$$

Here it is assumed that $x = z/R$, $y = r/R$. We see from (8) that for short lenses, given sufficiently large Reynolds numbers, we can treat the temperature, in approximate terms, as an exclusive function of the y coordinate. If we restrict ourselves to an investigation of the beams propagating in a region sufficiently close to the axis of the tube, instead of (8) we will have

$$T - T_0 = \frac{qR}{\lambda}\left(y^2 - \frac{7}{24}\right). \tag{9}$$

The dielectric constant is associated with temperature by the relationship

$$\varepsilon = 1 + \frac{\varepsilon_0 - 1}{T} T_0. \tag{10}$$

If $(T - T_0/T_0)^2 \ll 1$, the following equation is valid:

$$\frac{T_0}{T} = 1 - \frac{T-T_0}{T_0}. \tag{11}$$

Substitution of (11) and (9) into (10) yields the quadratic medium

$$\varepsilon = \varepsilon_0' - \alpha y^2, \tag{12}$$

where

$$\varepsilon_0' = \varepsilon_0 + \frac{7}{24}\frac{qR}{\lambda}\frac{(\varepsilon_0 - 1)}{T_0};\qquad \alpha = \frac{qR}{\lambda}\frac{(\varepsilon_0-1)}{T_0}. \tag{13}$$

Since $n = \sqrt{\varepsilon}$, according to (6) we have

$$x - c_2 = \int \frac{dy}{\sqrt{c_1(\varepsilon_0' - \alpha y^2) - 1}}. \tag{14}$$

After integration of (14) and simple transformations we have

$$y = \sqrt{\frac{c_1\varepsilon_0' - 1}{c_1\alpha}} \sin \sqrt{c_1\alpha}\,(x - c_2), \tag{15}$$

where

$$c_1 = \frac{y_0'^2 + 1}{\varepsilon_0' - \alpha y_0^2};\qquad c_2 = -\frac{1}{\sqrt{c_1\alpha}} \arcsin y_0 \sqrt{\frac{c_1\alpha}{c_1\varepsilon_0' - 1}}. \tag{16}$$

Thus, in this case the trajectory is a sinusoid with the amplitude

$$A = \sqrt{\frac{c_1\varepsilon_0' - 1}{c_1\alpha}} = \sqrt{\frac{\varepsilon_0'}{\alpha} - \frac{\varepsilon_0' - \alpha y_0^2}{\alpha(y_0'^2 + 1)}} \tag{17}$$

and the period

$$\tau = \frac{2\pi}{\sqrt{\alpha}}\frac{\sqrt{\varepsilon_0' - \alpha y_0^2}}{\sqrt{y_0'^2 + 1}}. \tag{18}$$

It follows from (17) and (18) that the amplitude and period of the oscillation are functions of y_0 and y_0'. A consequence of the relationship between the period and the boundary values is aberration, primarily spherical aberration. It is interesting to note that from the approximate solution of the trajectory equation for the quadratic medium the conclusion is drawn in [3] that the latter is free of aberrations.

If we do not limit ourselves to a region sufficiently close to the axis of the tube, in analogous fashion instead of (14) we find the beam trajectory which is expressed in terms of the following elliptical integral:

$$x - c_2 = \int \frac{dy}{\sqrt{c_1(\varepsilon_0' - \alpha y^2 + \frac{\alpha}{4} y^4) - 1}}, \tag{19}$$

where

$$c_1 = \frac{y_0'^2 + 1}{\varepsilon_0' - \alpha y_0^2 + \frac{\alpha}{4} y_0^4}. \tag{20}$$

If $c_1\varepsilon_0' < 1$, the integral in (19) is brought to the form

$$x - c_2 = \frac{2}{\sqrt{c_1\alpha}} \int \frac{dy}{\sqrt{(y^2 + a^2)(y^2 - b^2)}}, \tag{21}$$

while if $c_1\varepsilon_0' > 1$, it is brought to the form

$$x - c_2 = \frac{2}{\sqrt{c_1\alpha}} \int \frac{dy}{\sqrt{(a_1^2 - y^2)(b_1^2 - y^2)}}. \tag{22}$$

For example, let us analyze the integral in (21). From a comparison of (19) and (21) we find that

$$a^2 = -2 + 2\sqrt{1 + \frac{1 - c_1\varepsilon_0'}{c_1\alpha}},$$
$$b^2 = 2 + 2\sqrt{1 + \frac{1 - c_1\varepsilon_0'}{c_1\alpha}}. \tag{23}$$

The substitutions which transform the elliptical integral in (21) to normal Legendre form are different for the various integration intervals [5]. If $\eta > b > 0$, then

$$\int_b^\eta \frac{dy}{\sqrt{(y^2 + a^2)(y^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\varphi, k), \tag{24}$$

where $F(\varphi, k)$ is an elliptical integral of the first kind;

$$\varphi = \arccos \frac{b}{\eta};$$
$$k = \frac{a}{\sqrt{a^2 + b^2}}. \tag{25}$$

Substitution of (24) into (22) yields

$$x - c_2 = \frac{2}{\sqrt{c_1\alpha}} \frac{1}{\sqrt{a^2 + b^2}} F(\varphi, k). \tag{26}$$

It would be possible to perform calculations of this type for any integration interval [5]. Considering the functional equations

$$F(-\varphi, k) = -F(\varphi, k),$$
$$F(m\pi \pm \varphi, k) = 2mK(k) \pm F(\varphi, k), \tag{27}$$

where $K(k) = F(\pi/2\ \ k)$, we conclude that the maximum deviation of the beam from the axis of the lightguide will be at the points

$$x_m = \frac{(2m \pm 1) K(k) \cdot 2}{\sqrt{c_1\alpha} \sqrt{a^2 + b^2}} + c_2, \tag{28}$$

and the beam will intersect the axis of the tube at

$$x_n = \frac{2nK(k)\cdot 2}{\sqrt{c_1\alpha}\sqrt{a^2+b^2}}, \tag{29}$$

with the oscillation period

$$\tau_1 = x_m - x_{m-2} = \frac{4K(k)\cdot 2}{\sqrt{c_1\alpha}\sqrt{a^2+b^2}} \tag{30}$$

as in (18) being a function of the boundary values y_0, y_0', and the magnitude of the heat flow.

In conclusion, it should be pointed out that the derived formula (6) makes it possible not only to determine the trajectory for any n = n(y), but to find lens-like media which ensure the specified beam trajectory, which on occasion is no less important.

NOTATION

n	is the refractive index;
ε	is the dielectric constant;
y	is the transverse coordinate;
x	is the longitudinal coordinate;
q	is the heat flow;
λ	is the coefficient of thermal conductivity;
T_0	is the temperature at the tube inlet;
T	is the temperature at the point where the profile is completely developed;
R	is the tube radius;
ε_0'	is the dielectric constant at the inlet to the segment with a completely developed temperature profile;
τ	is the oscillation period;
$F(\varphi, k)$	is an elliptical integral of the first kind;
K(k)	is the total elliptical integral of the first kind.

LITERATURE CITED

1. D. W. Berreman, Journ. of Optical Society of America, 55, No. 3 (1965).
2. D. Marcuse, B.S.T.I., Vol. XLIV, 2087 (1965).
3. D. Marcuse, IEEE, v.MTT-13, No. 6 (1965).
4. S. E. Miller and D. Marcuse, B.S.T.I., 47, July (1964).
5. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, Moscow (1962).
6. R. Siegel, E. M. Sparrow, and T. M. Hallman, Appl. Sci. Res., Section A, 7, 386-392 (1958).
7. S. Goldstein, Modern Developments in Fluid Dynamics (1938), p. 622.